Chapter 7 Thick Disks and Accretion Flows

1. Thick disks
The standard thin disk is inconsistent with the super-Eddington accretion.

In the accretion disk the radiative force cannot be greater than the effective gravity force, and at the surface of the standard disk this condition yields,

\[ F_{\text{grav}} = \frac{G M m_p}{r^2} \left( \frac{H}{r} \right) > F_{\text{rad}} = \frac{\sigma T}{c} f_{\text{rad}} = \frac{\sigma T}{c} \frac{3GM \dot{M}}{8 \pi r^3} \left[ 1 - \left( \frac{3r_c}{r} \right) \right]^{\frac{1}{2}} \]
and
\[(H/r)_{max} > \dot{M}/\dot{M}_{\text{Edd}}.\]

This means that for a super-Eddington accretion with \(\dot{M}/\dot{M}_{\text{Edd}} > 1\) it also must be \(H/r > 1\).
Thick disks \((H \sim R)\) may be relevant to the following astrophysical phenomena.

The central power sources in radio galaxies and quasars;

Early stage of star formation, with bipolar outflows and jets;

Merger of two stars, etc.
Equilibrium and structure of fluids in pure rotation

We consider a hypothetical, steady thick disk, with no viscosity, in a state of *pure* rotation.

The implicit assumption is that the radial and vertical velocity components are small compared with velocity of rotation (it is not clear to date whether this assumption is justified), or in the cylindrical coordinates \((R, \phi, z)\) with the \(z\)-axis coincident with the rotation axis,

\[
\nu_R = 0, \quad \nu_\phi = R \Omega, \quad \nu_z = 0
\]

where \(\Omega = \Omega(R, z)\).
The equations of motion then reduce to the following

\[
\frac{1}{\rho} \frac{\partial P}{\partial R} = -\frac{\partial \Phi}{\partial R} + R\Omega^2, \quad \frac{1}{\rho} \frac{\partial P}{\partial z} = -\frac{\partial \Phi}{\partial z}
\]

where \( P \) and \( \Phi \) are the pressure and gravitational potential, respectively.

The equations can also be written in the vector form

\[
\frac{1}{\rho} \nabla P = -\nabla \Phi + \vec{R} \Omega^2 = \vec{g}_{\text{eff}}
\]

where the effective gravity \( g_{\text{eff}} \) is the sum of gravitational and centrifugal acceleration.
The effective gravity must be orthogonal to surfaces of constant pressure (*isobaric surfaces*). Take the curl of the above equation, we get

$$\nabla \times \left( \frac{1}{\rho} \nabla P \right) = -\frac{1}{\rho^2} (\nabla \rho \times \nabla P) = 2\Omega \nabla \Omega \times \vec{R} = 2 \frac{\partial \Omega}{\partial z} \vec{v}$$

The vectors $\nabla (1/\rho)$ and $\nabla P$ are orthogonal to isopycnic (constant density) and isobaric surfaces respectively.
If $\partial \Omega / \partial z = 0$, these vectors must be parallel everywhere, which implies that surfaces of constant pressure and constant density coincide.

When $\partial \Omega / \partial z = 0$ it is possible to introduce a rotational potential $\psi_{\text{rot}}$

$$\nabla \psi_{\text{rot}} = \vec{R} \Omega^2 (R)$$

and an effective potential $\Phi_{\text{eff}} = \Phi - \psi_{\text{rot}}$, so that

$$\vec{g}_{\text{eff}} = -\nabla \Phi_{\text{eff}}.$$
The equation for isobaric/isopycnic surfaces becomes simply
\[ \Phi_{\text{eff}} = \Phi(R, z) - \psi_{\text{rot}}(R) = \text{constant}. \]

Thus in a barotropic configuration the surfaces of constant pressure, density, and effective potential all coincide.
The shape of the isobaric surface follows from the condition
\[ \frac{dP}{dR} dR + \frac{\partial P}{\partial z} dz = 0, \]
which yields
\[ \frac{dz}{dR} = -\frac{(\partial P / \partial R)}{(\partial P / \partial z)} = -\frac{\partial \Phi / \partial R - R\Omega^2}{\partial \Phi / \partial z}. \]
From this equation one can solve for \( z(R) \) on an isobaric surface if \( \Phi \) and \( \Omega \) are known functions of \( z \) and \( R \).
Example:
Consider a disk the surfaces of which are generated by revolution
of two straight lines at angles ±α to the equatorial plane.

We assume that the disk has negligible mass, in the gravitational
field of a point of mass $M$ at the origin.

The equation for the cross-section of the disk is
\[ z_s(R) = \pm R \tan \alpha \]
Assuming $\Phi_{\text{eff}} = 0$ for the surface $z_s(R)$, we then have

$$\psi_{\text{rot}}(R) = \frac{GM \cos \alpha}{R}$$

$$\Omega^2(R) = \frac{1}{R} \frac{\partial \Psi_{\text{rot}}}{\partial R} = \frac{GM \cos \alpha}{R^3}$$

and the expression for equipotentials

$$r = \left[ GM / (-\Phi_{\text{eff}}) \right] (1 - \cos \alpha / \cos \lambda)$$

where $\lambda$ is the latitude.
The figure shows the meridional section of the equipotentials for the disk. Contours are labeled by energy values in arbitrary units. Only shown are negative values of permit disks of finite extent (shaded area).
● The limiting luminosity
For thick disks the luminosity can exceed the Eddington limiting luminosity by a large factor.

The reason is that in the funnel regions the centrifugal forces always exceed gravity and it is mainly the balance between pressure gradients and rotation that determines the equilibrium.

The maximum radiative output in these funnels is related to rotation rather than to the central mass and therefore can exceed $L_{\text{Edd}}$. 
For a stationary thick disk, the maximum flux that can be emitted from the disk surface is achieved when the effective gravity is balanced with the gradients of radiation pressure alone,

\[ \overrightarrow{F} = -\frac{c}{\kappa \rho} \nabla P = -\frac{c}{\kappa} \overrightarrow{g}_{\text{eff}} = \frac{c}{\kappa} (\nabla \Phi - \overrightarrow{R} \Omega^2) \]

Integrating \( F \) over the entire surface of the disk and assuming \( \kappa = \text{constant} \), we get the maximum luminosity
\[ L_{\text{max}} = \frac{c}{\kappa} \int_S \nabla \Phi \cdot dS - \frac{c}{\kappa} \int_S \Omega^2 \vec{R} \cdot dS = \frac{c}{\kappa} \int_V \nabla^2 \Phi \, dV - \frac{c}{\kappa} \int_V \nabla \cdot (\Omega^2 \vec{R}) \, dV \]

\[ = \frac{c}{\kappa} \int_V 4\pi G \rho \, dV - \frac{c}{\kappa} \int_V \frac{1}{2} \left\{ \frac{1}{R} \frac{\partial}{\partial R} (\Omega^2 R)^2 - \left( R \frac{\partial \Omega}{\partial R} \right)^2 \right\} \, dV \]

\[ = \frac{4\pi G M c}{\kappa} + \frac{c}{2\kappa} \int_V \left( R \frac{\partial \Omega}{\partial R} \right)^2 \, dV - \frac{c}{2\kappa} \int_V \left[ \frac{1}{R} \frac{\partial}{\partial R} (R^2 \Omega) \right]^2 \, dV \]

where \( M \) is the total mass enclosed by the surface \( S \) and not the central object. It is the second term, containing the shear, which makes the dominant contribution. The third term always reduces \( L_{\text{max}} \) unless it is vorticity-free (\( R^2 \Omega = \text{constant} \)).
The maximum luminosity that emitted between $R_1$ and $R_2$ by the disk given as an example before is

$$L_{\text{max}} = 2\int_{R_1}^{R_2} F_{\text{max}} (2\pi R / \cos \alpha) dR$$

where

$$F_{\text{max}} = \frac{c}{\kappa} g_{\text{eff}} = \frac{c}{\kappa} \frac{GM}{r^2} \tan \alpha = \frac{c}{\kappa} \frac{GM}{R^2} \sin \alpha \cos \alpha$$

Then

$$L_{\text{max}} = L_{\text{Edd}} \sin \alpha \ln(R_2/R_1).$$

Clearly $L_{\text{max}}$ can exceed $L_{\text{Edd}}$ by a large factor if $R_2/R_1$ is sufficiently large. It arises because of the large radiative pressure gradients necessary to support the walls of the funnel.
However, the logarithm in above is of a crucial importance, as it prevents astrophysically realistic doughnuts (i.e. with $r_{\text{out}}/r_0 < 10^6$, say) to have highly super-Eddington luminosities.

Thus, the theory predicts that for such “realistic” fat tori, only a slightly super-Eddington total (isotropic) luminosities,

$$L_{\text{total}}/L_{\text{Edd}} \sim \ln\left( \dot{M}/\dot{M}_{\text{Edd}} \right),$$

may be expected.
● Thick accretion disks
There are two main differences between thick and thin disks.

(i) Since in thick disks the rotation field is not uniquely determined by the balance with gravity, we are free to choose the specific angular momentum distribution.

(ii) We cannot assume local energy balance, because the heat generated by dissipation can travel in any direction before emerging to the surface. So in steady state the total luminosity $L$ must equal the total rate of energy generation by viscosity $L_{\text{gen}}$. 
The total rate of dissipation in a cylindrical shell between radii \( R \) and \( R + dR \) is
\[
dL_{\text{gen}} = G(R)(d\Omega/dR)dR
\]
and
\[
G(R) = G_0 - \dot{M}(l - l_0)
\]
where \( G_0 \) and \( l_0 \) are the torque and specific angular momentum at some arbitrary radius.
The total luminosity generated between radii $R_1$ and $R_2$ is

$$L_{\text{gen}}(R_1, R_2) = \int_{R_1}^{R_2} [G_0 - \dot{M}(l - l_0)](d\Omega / dR)dR$$

$$= G_0(\Omega_2 - \Omega_1) + \dot{M}l_0(\Omega_2 - \Omega_1) - \dot{M}[(\nu_2^2 / 2 - \nu_1^2 / 2) - (\Psi_{\text{rot},2} - \Psi_{\text{rot},1})]$$

$$= G_0(\Omega_2 - \Omega_1) + \dot{M}l_0(\Omega_2 - \Omega_1) - \dot{M}[(\nu_2^2 / 2 - \nu_1^2 / 2) - (\Phi_{\text{rot},2} - \Phi_{\text{rot},1})]$$

Introducing the energy per unit mass $e = \nu^2/2 + \Phi$ and using $\nu^2 = \Omega l$, we apply this expression to the whole disk as

$$L_{\text{gen}} = L_{\text{gen}}(R_{\text{in}}, R_{\text{out}}) = G_{\text{in}}(\Omega_{\text{out}} - \Omega_{\text{in}}) + \dot{M}[(e_{\text{out}} - e_{\text{in}}) - \Omega_{\text{out}}(l_{\text{out}} - l_{\text{in}})]$$
Assume $G_{in} = 0$, $l_{out} \Omega_{out} \rightarrow 0$ as $R_{out} \rightarrow \infty$, for $R_{out} \gg R_{in}$, one has

$$L_{gen} = \dot{M}(-e_{in}) = \frac{GMM}{2R_{in}} \left( 2 - \frac{l_{in}^2}{GMR_{in}} \right).$$

So if the disk is Keplerian near its inner radius, the luminosity is equal to the standard value.

Note that the above results will have to be modified for a thick disk with $R_{in} \sim R_S$, since the Newtonian potential does not adequately describe the gravitational field near a black hole.
2. Advection dominated accretion flows (ADAFs)


The specific abbreviation ADAF, which stands for advection-dominated accretion flow, was introduced by Lasota (1996, in Physics of Accretion Disks).
As the luminosity of an accreting black hole drops to a few percent of Eddington, the spectrum switches from the familiar soft state to a hard state that is well-described by an ADAF.

An ADAF is a poor radiator, and the ion temperature can approach $10^{12}$ K near the center, although the electrons are cooler, with temperature $\sim 10^{9-11}$ K.

The accreting gas carries the bulk of the accretion energy as stored thermal energy, which vanishes without a trace as the gas passes through the hole’s event horizon. The large thermal energy in an ADAF would drive strong winds and jets.
The fact that black holes possess an event horizon instead of a hard surface makes the inner boundary condition for black hole accretion flows qualitatively different from that for an external distant observer.

ADAFs onto black holes are characterized by a low radiative efficiency $\eta$, while in ADAF onto objects with a hard surface the advected energy must be ultimately reprocessed and released near the surface.
The figure shows a comparison between black hole (BH, filled circles) and neutron star (NS, open circles) SXT luminosity variations.

The ratio of the quiescent luminosity to the peak outburst luminosity is systematically smaller for BH systems than for NS systems. This indicates the presence of event horizons at the center of BH candidate systems (see however, Fender et al. 2003, MNRAS, 343, L99).
Low radiative efficiencies may occur in the following two cases

(1) At super-Eddington accretion rates, the large optical depth of the inflowing gas traps most of the radiation and carries it inward, or “advects” it, into the central black hole.

\[ t_{\text{diff}} (\text{diffusion time for photons}) \gg t_{\text{acc}} (\text{accretion time}). \]

This solution is referred to as an optically thick ADAF, or the “slim disk” model developed by Abramowicz et al. (1988).

The black hole may accrete at well above the Eddington critical rate and yet produce a sub-Eddington luminosity.
(2) At low, sub-Eddington accretion rates, the accreting gas has a very low density and is unable to cool efficiently within an accretion time.

The viscous energy is therefore stored in the gas as thermal energy instead of being radiated, and is advected onto the central star.

\[ t_{\text{cool}} \text{ (cooling time)} \gg t_{\text{acc}} \text{ (accretion time)} \]

The gas is optically thin and adopts a two-temperature configuration. The solution is therefore referred to as an optically thin ADAF or a two-temperature ADAF, sometimes referred to as a RIAF — a “radiatively inefficient accretion flow”.
Dynamics of ADAFs
(1) Basic equations
The dynamics of a steady axisymmetric accretion flow are described by the following four height-integrated differential equations:

\[ \frac{d}{dR} (\rho R H v) = 0 \]

\[ \nu \frac{dv}{dR} - \Omega^2 R = -\Omega_K^2 R - \frac{1}{\rho} \frac{d}{dR} (\rho c_s^2) \]

\[ \nu \frac{d(\Omega R^2)}{dR} = \frac{1}{\rho H} \frac{d}{dR} \left( \nu \rho R^3 H \frac{d\Omega}{dR} \right) \]

\[ q^{\text{adv}} = \rho v T \frac{ds}{dR} = q^+ - q^- = \rho v R^2 \left( \frac{d\Omega}{dR} \right)^2 - q^- = f \rho v R^2 \left( \frac{d\Omega}{dR} \right)^2 \]
where $s$ is the specific entropy of the gas, $q^+$ is the energy generated by viscosity per unit volume, and $q^-$ is the radiative cooling per unit volume.

The parameter $f$ is the ratio of the advected energy to the heat generated and measures the degree to which the flow is advection-dominated.

The kinematic viscosity coefficient $\nu$ is generally parameterized via the $\alpha$ prescription of Shakura & Sunyaev (1973)

$$\nu = \alpha c_s H$$
Depending on the relative magnitudes of the terms in the energy equation, three regimes of accretion may be identified:

(i) $q^+ \approx q^- >> q^{\text{adv}}$. This corresponds to a cooling-dominated flow where all the energy released by viscous stresses is radiated; the amount of energy advected is negligible. The thin disk solution and the SLE solution correspond to this regime.

$L \sim 0.1 M c^2$
(ii) \( q^{adv} \approx q^+ >> q^- \). This corresponds to an ADAF where almost all the viscous energy is stored in the gas and is deposited into the black hole. The amount of cooling is negligible compared with the heating. For a given \( \dot{M} \), an ADAF is much less luminous than a cooling-dominated flow.
\[ L << 0.1\dot{M}c^2 \]

(iii) \( -q^{adv} \approx q^- >> q^+ \). This corresponds to a flow where energy generation is negligible, but the entropy of the inflowing gas is converted to radiation. Examples are Bondi accretion, Kelvin-Helmholtz contraction during the formation of a star, and cooling flows in galaxy clusters.
(2) Self-similar solution
Narayan & Yi (1994, ApJ, 428, L13) have derived the self-similar solution to the structure of optically thick and optically thin ADAFs by assuming Newtonian gravity (and taking $f$ to be independent of $R$),

$$v(R) = -\frac{(5 + 2 \varepsilon')}{3\alpha^2} g(\alpha, \varepsilon') \alpha v_{ff}$$

$$\Omega(R) = \sqrt{\frac{2 \varepsilon'(5 + 2 \varepsilon')}{9\alpha^2} g(\alpha, \varepsilon')} \frac{v_{ff}}{R}$$

$$c_s^2(R) = \frac{2(5 + 2 \varepsilon')}{9\alpha^2} g(\alpha, \varepsilon') v_{ff}$$
where
\[
\nu_{ff} = \left(\frac{GM}{R}\right)^{1/2}, \quad \varepsilon' = \frac{\varepsilon}{f'} = \frac{1}{f'} \left(\frac{5/3 - \gamma}{\gamma - 1}\right), \quad g(\alpha, \varepsilon') = \left[1 + \frac{18\alpha^2}{(5 + 2\varepsilon')^2}\right]^{1/2} - 1
\]

\(\gamma\) is the ratio of specific heats of the gas, which is likely to lie in the range 4/3 to 5/3 (the two limits correspond to a radiation pressure dominated and a gas pressure dominated accretion flow, respectively).

Correspondingly, \(\varepsilon\) lies in the range 0 to 1.
In general, $f$ depends on the details of the heating and cooling and will vary with $R$. The assumption of a constant $f$ is therefore an oversimplification.

However, when the flow is highly advection dominated, $f \sim 1$ throughout the flow, and can be well approximated as constant.

Setting $f = 1$ and taking the limit $\alpha^2 \ll 1$ (which is nearly always true), the solution takes the simple form

$$\frac{\nu}{v_{ff}} \approx -\left(\frac{\gamma - 1}{\gamma - 5/9}\right)\alpha, \quad \frac{\Omega}{\Omega_K} \approx \left[\frac{2(5/3 - \gamma)}{3(\gamma - 5/9)}\right]^{1/2}, \quad \frac{c_s^2}{v_{ff}^2} \approx \frac{2}{3} \left(\frac{\gamma - 1}{\gamma - 5/9}\right).$$
Figure 1. Angular profiles for radial self-similar solutions with $\alpha = 0.1, \epsilon' = 0.1, 1, 10$. Top left: angular velocity $\Omega/\Omega_K$ as a function of polar angle $\theta$. Top right: radial velocity, $v/v_R$. Bottom left: density, $\rho$. Bottom right: sound speed squared, $c_s^2/v_R^2$.

From Narayan et al. astro-ph/9803141
A number of interesting features of ADAFs are revealed by the self-similar solution.

(i) It appears that ADAFs have relatively large values of the viscosity parameter, $\alpha \leq 1$; typically, $\alpha \sim 0.2-0.3$. This means that the radial velocity of the gas in an ADAF is comparable to the free-fall velocity, $v \leq v_{\text{ff}}$.

(ii) The gas rotates with a sub-Keplerian angular velocity and is only partially supported by centrifugal forces. In the extreme case when $\gamma \rightarrow 5/3$, the flow has no rotation at all ($\Omega \rightarrow 0$).
(iii) Since most of the viscously generated energy is stored in the gas as internal energy, rather than being radiated, the gas temperature is quite high; in fact, optically thin ADAFs have almost virial temperatures. This causes the gas to “puff up”: $H \sim c_s/\Omega \sim v_{ff}/\Omega_K \sim R$.

Therefore, geometrically, ADAFs resemble spherical Bondi (1952) accretion more than thin disk accretion. It is however, important to note that the dynamics of ADAFs are very different from that of Bondi accretion\(^1\).

\(^1\) Geometrically, ADAFs are similar to spherical Bondi accretion. They are quasi--spherical and have radial velocities which are close to free fall, at least for large $\alpha$. Also, the gas passes through a sonic radius and falls supersonically into the black hole.
Despite these similarities, it is important to stress that ADAFs are *dynamically* very different from pure spherical accretion. Transport of angular momentum through viscosity plays a crucial role in ADAFs; indeed, there would be no accretion at all without viscosity. Bondi accretion, on the other hand, involves only a competition between inward gravity and outward pressure gradient; viscosity plays no role.

In the Bondi solution, the location of the sonic radius is determined by the properties of the accreting gas at large radii. Under many conditions, the sonic radius is quite far from the black hole. In an ADAF, on the other hand, the sonic radius is almost always at a few $R_s$. Furthermore, the location of the sonic radius depends on the viscosity parameter $\alpha$ (another indication of the importance of viscosity) and the properties of the gas at infinity are irrelevant so long as the gas has enough angular momentum to prevent direct radial infall. In addition, the gas temperature, or energy density, in an ADAF is determined by viscous heating and adiabatic compression, and is generally much higher than in Bondi accretion. Also, the rotational velocities in ADAFs are quite large, reaching nearly virial velocities, whereas $\Omega$ is strictly zero in the Bondi solution.

In nature, when gas accretes onto a black hole, it nearly always has too much angular momentum to fall directly into the hole. ADAFs are, therefore, much more relevant than the Bondi solution for the description of accretion flows around black holes.
Numerical calculations confirm that ADAFs are quasi-spherical and quite unlike thin disks. The global solutions agree quite well with the self-similar solutions, except near the boundaries, where there are significant deviations\(^2\). This means that the self-similar solution provides a good approximation to the real solution over most of the flow.

\(^2\) The boundary conditions are as follows. Far from a central black hole, an ADAF might join on to a thin disk. At this radius, the rotational and radial velocities, as well as the gas density and sound speed, must take on values appropriate to a thin disk. Also, as the accreting gas flows in toward the black hole, it must undergo a sonic transition at some radius, where the radial velocity equals the local sound speed. In addition, since the black hole cannot support a shear stress, the torque at the horizon must be zero.
Two-temperature ADAFs
(1) Basic assumptions
(i) Equipartition magnetic fields
There is strict equipartition between gas and (tangled) magnetic pressure, i.e.

\[
p_m = \frac{B^2}{24\pi} = (1 - \beta) \rho c_s^2
\]

where \(\beta = 0.5\). Then \(\gamma = (8 - 3\beta)/(6 - 3\beta)\).
(ii) Thermal coupling between ions and electrons
Ions and electrons interact only through Coulomb collisions and there is no non-thermal coupling between the two species. In this case the plasma is two temperature, with the ions much hotter than the electrons.

(iii) Preferential heating of ions
Most of the turbulent viscous energy goes into the ions, and only a small fraction $\delta << 1$ goes into the electrons.

(iv) $\alpha$ viscosity
The constant viscosity parameter $\alpha$ of Shakura & Sunyaev (1973) is used to described angular momentum transport.
(2) Properties of two-temperature, optically-thin ADAFs

The scaling laws of various quantities in an ADAF as a function of the model parameters with \( f \to 1 \) and \( \beta = 0.5 \) are

\[
\begin{align*}
\nu & \approx -1.1 \times 10^{10} \alpha r^{-1/2} \text{ cm s}^{-1}, \\
\Omega & \approx 2.9 \times 10^{4} \alpha m^{-1} r^{-3/2} \text{ s}, \\
c_s^2 & \approx 1.4 \times 10^{20} \alpha r^{-1} \text{ cm}^2 \text{s}^{-2}, \\
n_e & \approx 6.3 \times 10^{19} \alpha^{-1} m^{-1} \dot{m} r^{-3/2} \text{ cm}^{-3}, \\
B & \approx 7.8 \times 10^{8} \alpha^{-1/2} m^{-1/2} \dot{m}^{1/2} r^{-5/4} \text{ G}, \\
p & \approx 1.7 \times 10^{16} \alpha^{-1} m^{-1} \dot{m} r^{-5/2} \text{ g cm}^{-1} \text{s}^{-2}, \\
qu^+ & \approx 5.0 \times 10^{21} m^{-2} \dot{m} r^{-4} \text{ erg cm}^{-3} \text{s}^{-1}, \\
\tau_{es} & \approx 24 \alpha^{-1} \dot{m} r^{-1/2},
\end{align*}
\]
where \( m = M/M_{\text{sun}} \), \( r = R/R_S \),

\[
\dot{m} = \frac{\dot{M}}{\dot{M}_{\text{Edd}}} \quad \text{and} \quad \dot{M}_{\text{Edd}} = 1.39 \times 10^{18} m \text{ gs}^{-1}.
\]

The optically thin ADAF solution exists only for \( \dot{m} < \dot{m}_{cr} \).

Theoretical investigations suggest \( \dot{m}_{cr} \propto \alpha^2 \) out to \( \sim 10^2-10^3 R_S \),

Observations suggest that the two-temperature ADAF solution exists up to \( \dot{m}_{cr} \sim 0.05-0.1 \). This suggests that \( \alpha \sim 0.2-0.3 \) in ADAFs.
(3) ADAF spectra and radiation processes

In a two-temperature ADAF the ions receive most of the viscous energy and are nearly virial,

\[ T_i \approx 2 \times 10^{12} \beta r^{-1} \text{ K.} \]

The electrons, on the other hand, are heated by several processes with varying efficiencies - Coulomb coupling with the ions, compression, and direct viscous heating - and cooled by a variety of radiation processes.
The electron energy equation is
\[ \rho v T_e \frac{ds}{dR} = \rho v \frac{d\varepsilon}{dR} - q^c = q^{ie} - q^- \]

where \( s \) and \( \varepsilon \) are the entropy and internal energy of the electrons per unit mass of the gas, and \( q^c \), \( q^{ie} \) and \( q^- \) are the compressive heating (or cooling) rate, the heating rate via Coulomb collisions with the hotter protons and the energy loss due to radiative cooling per unit volume. Viscous heating is neglected here.
When $\dot{m} \gtrsim 0.1 \alpha^2$, $q^{\text{ie}} \gg q^c$, the energy equation becomes

$$\rho v \frac{d\varepsilon}{dR} \simeq q^{\text{ie}} + q^c - q^- \simeq q^{\text{ie}} - q^-$$

Since the cooling is efficient enough to radiate away all of the energy given to the electrons, $q^{\text{ie}} = q^-$ for $r < \sim 10^2$, and the internal energy of the electrons does not change with radius, $d\varepsilon/dR = 0$.

The electron temperature therefore remains essentially constant at $\sim 10^9$ K.
When \( \dot{m} \lesssim 10^{-4} \alpha^2, \ q^{ie} \ll q^c \)

\[
\rho v \frac{d\varepsilon}{dR} \approx q^c + q^{ie} - q^- \approx q^c
\]

The radial dependence of the electron temperature is determined by adiabatic compression. The electron temperatures in this regime are slightly higher.

For \( 10^{-4} \alpha^2 \leq \dot{m} \leq 0.1 \alpha^2 \), the electron temperature profile lies in between these two extremes.
The figure shows the temperature profile as a function of radius for various $\dot{m}$.

The electrons become cooler as the accretion rate increases, because at high $\dot{m}$, the dominant cooling mechanism is inverse Compton scattering, which is an extremely efficient process when the optical depth approaches unity and its efficiency increases sharply with increasing optical depth.

Figure 4. Variation of electron temperature $T_e$ with radius $r$ for ADAFs with (from top to bottom) log($\dot{m}$) = −2, −1.8, −1.6, −1.4, −1.2, −1.1 (taken from Esin et al. 1997). Note that the electron temperature decreases with increasing $\dot{m}$. 
The spectrum from an ADAF around a black hole ranges from radio frequencies to gamma-ray frequencies, and can be divided into two parts based on the emitting particles:

(a) The radio to hard X-ray radiation is produced by electrons via synchrotron, bremsstrahlung and inverse Compton processes.

(b) The gamma-ray radiation results from the decay of neutral pions created in proton-proton collisions.\(^3\)

\(^3\) Although the electrons achieve relativistic temperatures, pair processes are found to be unimportant in two-temperature ADAFs because of the low density, which allows very few pair-producing interactions in the medium.
The figure shows schematically the various elements in the spectrum of an ADAF around a black hole. S, C, and B refer to electron emission by synchrotron radiation, inverse Compton scattering, and bremsstrahlung, respectively. The solid line corresponds to a low $\dot{m}$, the dashed line to an intermediate $\dot{m}$, and the dotted line to a high $\dot{m} \sim \dot{m}_{cr}$. The $\gamma$-ray spectrum is due to the decay of neutral pions created in proton-proton collisions.
The figure shows four sequences of spectra, two of which corresponding to ADAFs (excluding the gamma-ray component) and the other two to thin disks.

(a) Spectra from an ADAF around a $10 \, M_{\text{sun}}$ black hole for (from top to bottom) 
$\log \dot{m} = -1.1, -1.5, -2, -2.5, -3, -3.5,$ -

(b) Spectra from a thin disk at the same accretion rates.

(c) and (d) show the corresponding spectra for a $10^9 \, M_{\text{sun}}$ black hole.
Note that these spectra are for pure disk and pure ADAF models. In practice, real systems are often modeled as an ADAF surrounded by a thin disk. In such composite models, the ADAF part of the spectrum is essentially unchanged, but a dimmer and softer version of the thin disk is also present in the spectrum.
Another striking feature seen in the figure is that ADAFs are much less luminous than thin disks at low values of $\dot{m}$. In fact, the luminosity of an ADAF scales roughly as $\sim \dot{m}^2$ rather than $\dot{m}$ for a thin disk.
• Transition radius
Narayan, McClintock, & Yi (1996, 457, 821) propose that the accretion flow consists of two zones separated at a transition radius $r_{tr}$. For $r < r_{tr}$, there is a two-temperature ADAF. For $r > r_{tr}$, the accretion occurs partially as a thin accretion disk, and partially as a hot corona, modeled as an ADAF.

The transition radius $r_{tr}$ is determined principally by $\dot{m}$, but the exact form of $r_{tr}(\dot{m})$ is not known. It could be the maximum $r$ out to which an ADAF is allowed for the given $\dot{m}$; equivalently, it is that $r$ at which $\dot{m} = \dot{m}_{cr}(r)$. 
Several proposals have been made to explain why the inflowing gas might switch from a thin disk to an ADAF at the transition radius. It is not clear which, if any, of these mechanisms is most important.

(1) Meyer & Meyer-Hofmeister (1994, A&A, 288, 175) proposed, for cataclysmic variables, a mechanism in which the disk is heated by electron conduction from a hot corona; this evaporates the disk, leading to a quasi-spherical hot accretion flow.
(2) Honma (1996, PASJ, 48, 77) suggested that turbulent diffusive heat transport from the inner regions of the ADAF produces a stable hot accretion flow out to large radii, which then joins to a cool thin disk.

(3) Narayan & Yi (1995, ApJ, 452, 710) suggested that small thermal instabilities in the optically-thin upper layers of a thin disk might cause the disk to switch to an ADAF.

For a recent discussion see Lu, Gu and Liu (2003, IAUS 214, 91).
• Application
ADAF models have been applied to a number of accreting black hole systems. They give a satisfying description of the spectral characteristics of several quiescent black hole binaries and low luminosity galactic nuclei which are known to experience low efficiency accretion.
The figure shows the spectrum of an ADAF model of A0620-00 (solid line) at an accretion rate of $\dot{m} = 4 \times 10^{-4}$, compared with the observational data.

The dashed line is an ADAF model with $\beta = 0.8$, instead of the standard value of $\beta = 0.5$. The dotted line shows the spectrum of a thin accretion disk with an accretion rate $\dot{m} = 1 \times 10^{-5}$, adjusted to fit the optical flux.
The figure shows quiescent spectrum of Sgr A*. The solid line is an ADAF model of Sgr A* in the quiescent state. The mass accretion rate is \( \dot{m} = 1 \times 10^{-6} \) near the BH. (From Yuan et al. 2003)
3. Advection-dominated inflow-outflow solutions (ADIOS)

In the ADAF solution the Bernoulli constant,

\[ Be \equiv \frac{1}{2} \Omega^2 r^2 - \frac{1}{r} + \frac{\gamma c_s^2}{\gamma - 1} = \Omega^2 r^2, \]

is necessarily positive. This implies that any exposed gas can escape to infinity with positive energy.
Blandford & Begelman (1999, MNRAS, 303, L1) assume that the mass inflow rate varies as
\[ \dot{m} \propto r^p, \]
with \(0 \leq p \leq 1\). The rotational and radial velocities scale as \(\propto r^{-1/2}\) and the density varies as \(\rho \propto r^{-3/2}\). With \(p=0\) (no wind) one recovers the standard ADAF solution.
The ADIOS solutions can also be used to fit the observations of a variety of astrophysical objects.

In general one would expect that the same observations which are successfully modeled by an ADAF can be also accommodated in an ADIOS model with higher mass supply rate.
Observational evidence for the association of nonthermal relativistic jets with ADAFs has accumulated in recent years with the discovery of radio emission in virtually every BHB in the hard state.

The radio emission is generally too bright to be produced by thermal electrons in the accretion flow. It is therefore very likely to come from nonthermal electrons in a jet.

(From R. Fender, astro-ph/0409464)
Thus, it is now observationally well-established that the hard state/ADAF is associated with relativistic jets.

A strong correlation has been seen between radio and X-ray luminosity in the hard state and quiescent state (Gallo et al. 2003).

Figure 2. The radio flux density ($S_{\text{radio}}$) is plotted against the X-ray flux density ($S_{\text{x-ray}}$) for a sample of 10 hard state BHs (see Table 1), scaled to a distance of 1 kpc and absorption corrected (this means that the axes are proportional to luminosities). On the top horizontal axis we indicate luminosity, in Eddington units for a 10-M_\odot BH, corresponding to the underlying X-ray flux density. An evident correlation between these two bands appears and holds over more than three orders of magnitude in luminosity. The dashed line indicates the best fit to the correlation, that is $S_{\text{radio}} = k(S_{\text{x-ray}})^{0.7}$, with $k = 223 \pm 156$ (obtained by fixing the slope at +0.7, as found individually for both GX 339–4 and V404 Cygni; see Section 4.1). Errors are given at the 3σ confidence level and arrows also represent 3σ upper limits.
4. Convection-dominated accretion flows (CDAFs)

Numerical simulations of ADAFs have shown that outflows can occur at rather large viscosities ($\alpha \geq 0.3$). At lower viscosities ($\alpha \leq 0.1$) the flows are convectively unstable developing large eddies.

Such CDAFs have a different structure than ADAFs. The angular momentum per unit volume $\rho R^2 \Omega$ is nearly constant as a result of the balance between the inward transport by convection and the outward transport by anomalous viscosity. The time-averaged radial density profiles are much flatter ($\rho \propto r^{-1/2}$) than the standard ADAF ($\rho \propto r^{-3/2}$).
References:
1. Frank, J., King, A. & Raine, D. 2002, Accretion power in astrophysics (CUP)
3. Abramowicz, M. A. astro-ph/0411185